

# Post-Newtonian Theory and its Application<sup>1</sup>

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## Abstract

We review some recent works on the post-Newtonian theory of slowly-moving (post-Newtonian) sources, and its application to the problems of dynamics and gravitational radiation from compact binary systems. Our current knowledge is 3PN on the center-of-mass energy and 3.5PN on the gravitational-wave flux of inspiralling compact binaries. We compute the innermost circular orbit (ICO) of binary black-hole systems and find a very good agreement with the result of numerical relativity. We argue that the gravitational dynamics of two bodies of comparable masses in general relativity does not resemble that of a test particle on a Schwarzschild background. This leads us to question the validity of some “Schwarzschild-like” templates for binary inspiral which are constructed from post-Newtonian resummation techniques.

## 1 Introduction

Recent years have shown a tremendous revival of interest in an Old Lady: the post-Newtonian approximation (or expansion when the speed of light  $c \rightarrow +\infty$ ), which is surely the most important technique in the arsenal of General Relativity for drawing firm predictions for the outcome of experimental facts related to gravitation. The post-Newtonian approximation is ideally suited for describing the adiabatic phase of the very interesting astrophysical systems known as inspiralling compact binaries. These systems constitute our only known-to-exist (for sure) source to hunt for in the current network of laser-interferometric detectors of gravitational waves, today composed of the large-scale interferometers VIRGO and LIGO, and the medium-scale ones GEO and TAMA.

Two compact (i.e. gravitationally-condensed) objects — neutron stars or black holes — orbit an inward spiral, with decreasing orbital radius  $r$ , decreasing orbital period  $P$ , and increasing orbital frequency  $\omega = \frac{2\pi}{P}$ . The inspiral is driven by the loss of energy associated with the gravitational-wave emission, or, equivalently, by the action of radiation forces. There is a long phase of *adiabatic* inspiral, with associated dimensionless adiabatic parameter

$$\frac{\dot{\omega}}{\omega^2} = \mathcal{O}\left(\frac{1}{c^5}\right). \quad (1)$$

The order of magnitude of the right-hand-side is that of the radiation-reaction force, namely  $\sim 1/c^5$  or equivalently 2.5PN order beyond the standard Newtonian acceleration (following the usual post-Newtonian jargon). The binary’s dynamics being essentially “aspherical”, inspiralling compact binaries are strong emitters of gravitational radiation.

The main point about the theoretical description of inspiralling compact binary is that a model made of two structureless point-particles, characterized solely by two mass parameters  $m_1$  and  $m_2$  (and possibly two spins), is sufficient. Most of the non-gravitational effects usually plaguing the dynamics of binary star systems: a magnetic field, an interstellar medium, etc., are dominated by gravitational forces. However, the real justification for a model of point particles is that the effects due to the finite size of the compact bodies are small. In particular, the tidal interactions between the two compact objects are expected to play a little role during most of the inspiral phase; the mass transfer (in the case of neutron stars) does not occur until very late, near the final coalescence. Thus the inspiralling compact binaries are very clean

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systems, essentially dominated by gravitational forces during most of their life. This is the reason why these systems are so interesting for relativity theorists!

Our fascination for the old post-Newtonian lady is that it was recognized that *improved* waveform modelling is crucial for constructing efficient templates for searching and measuring the gravitational waves from inspiralling compact binaries in the LIGO/.../TAMA network. By *improved* we mean in fact *much* improved with respect to what is known from the “Newtonian” approximation, which corresponds in the case of the radiation field to the Einstein quadrupole formalism. And one needs much improved modelling because the orbital motion of the inspiralling compact binary — which is responsible for the gravitational-wave emission — becomes *very* relativistic when the binary reaches the so-called Innermost Circular Orbit or ICO (defined below in this article). At that point the orbital velocity is of the order of 50% of the speed of light. The price to be paid for applying the post-Newtonian approximation is that we must go to very *high* post-Newtonian order. The status of the field nowadays is the 3PN approximation (i.e.  $\sim 1/c^6$ ) — or even better the 3.5PN one —, which is likely to be sufficient for practical purpose (at least in the case of neutron stars binaries).

After the two objects have passed the ICO they will plunge together and merge to form a single black hole, which will subsequently settle down into a stationary configuration, by emission of gravitational waves in the quasi-normal mode channel. Of course we should not expect the post-Newtonian approximation (and the description of the compact objects by point-particles) to be valid after the ICO, and we shall have to replace this model by a fully relativistic numerical computation of the phase of plunge and merger of two black holes [1]. But, up to the point of the ICO, the post-Newtonian approximation, when carried out to 3PN order, *is* physically valid, and is probably very accurate. This last point may appear to be a little surprising (it contradicts some statements in the literature about the “failure” of the post-Newtonian approximation in the regime of the ICO), but we shall present arguments supporting it. In particular the 3PN order is probably able to locate the ICO to within an accuracy of the order of 1% or better (for binary systems with comparable masses).

## 2 Post-Newtonian iteration for isolated systems

Before embarking into strong statements concerning the “accuracy” of the post-Newtonian description of binary systems, it is wise to examine in a somewhat general way the well-definiteness of the approximation itself. This is especially important in view of the fact that our old lady the post-Newtonian approximation has been unjustly accused of being plagued with some apparently inherent difficulties, and furthermore which crop up around the 3PN order we are interested in. The problems arise in the general case of regular (singularity-free) matter sources. Up to the 2.5PN order the approximation can be worked out without problems, and often at the 3PN order the problems can be solved specifically for each case at hands. However, it must be admitted that these difficulties, even if appearing at higher approximations, have cast doubt in the past on the actual soundness, on the theoretical point of view, of the post-Newtonian expansion. They pose the practical question of the reliability of the approximation when comparing the theory’s predictions with very precise experimental results. Here we discuss the nature of the problems — are they purely technical or linked with some fundamental drawback of the approximation? — and outline their resolution recently proposed in Ref. [2].

The first problem is that in higher approximations some *divergent* Poisson-type integrals appear. Recall that the post-Newtonian expansion replaces the resolution of an hyperbolic-like d’Alembertian equation by a perturbatively equivalent hierarchy of elliptic-like Poisson equations. Rapidly it is found during the post-Newtonian iteration that the right-hand-side of the Poisson equations acquires a non-compact support (it is distributed over all space), and that the standard Poisson integral diverges because of the bound of the integral at spatial infinity, i.e.  $r \equiv |\mathbf{x}| \rightarrow +\infty$ , with  $t = \text{const}$ . For instance some of the potentials occurring at the 2PN order in Chandrasekhar’s work [3] are divergent, so the corresponding metric is formally infinite. In fact, Kerlick [4, 5] showed that the post-Newtonian computation *à la* Chandrasekhar, following the iteration scheme of Anderson and DeCanio [6], can be made well-defined up to the 2.5PN order, but that this does not solve the problem at the next 3PN order, which has been found to involve some inexorably divergent Poisson integrals.

These divergencies come from the fact that the post-Newtonian expansion is actually a singular perturbation, in the sense that the coefficients of the successive powers of  $1/c$  are not uniformly valid

in space, since they typically blow up at spatial infinity like some positive powers of  $r$ . For instance, Rendall [7] has shown that the post-Newtonian expansion cannot be “asymptotically flat” starting at the 2PN or 3PN level, depending on the adopted coordinate system. The result is that the Poisson integrals are in general badly-behaving at infinity. Trying to solve the post-Newtonian equations by means of the standard Poisson integral does not *a priori* make sense. This does not mean that there are no solution to the problem, but simply that the Poisson integral does not constitute the correct solution of the Poisson equation in the context of post-Newtonian expansions. So the difficulty is purely of a technical nature, and will be solved once we succeed in finding the appropriate solution to the Poisson equation<sup>3</sup>.

To cure the problem of divergencies we have introduced [2], at any post-Newtonian order, a generalized solution of the Poisson equation with non-compact support source, in the form of an appropriate *finite part* of the usual Poisson integral: namely we regularize the bound at infinity of the Poisson integral by means of a process of analytic continuation, analogous to the one already used to regularize the retarded integrals in Refs. [8, 12, 9, 10]. At any post-Newtonian order  $n$  we have to solve a Poisson equation with non-compact-support “source-term”  $\bar{\tau}_n^{\mu\nu}$ , where the overbar indicates that we are considering a (formal) post-Newtonian expansion, and pick up the coefficient of  $1/c^n$  in that expansion ( $\mu, \nu$  are space-time indices). We multiply the source-term by a regularization factor  $r^B$ , where  $r = |\mathbf{x}|$  and  $B \in \mathbb{C}$ , and then apply the standard Poisson integral. The result,

$$\Delta^{-1} [r^B \bar{\tau}_n^{\mu\nu}] (\mathbf{x}, t) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{d^3\mathbf{y}}{|\mathbf{x} - \mathbf{y}|} |\mathbf{y}|^B \bar{\tau}_n^{\mu\nu}(\mathbf{y}, t), \quad (2)$$

is a well-defined integral (i.e. convergent at infinity,  $r \rightarrow +\infty$ ) in some region of the  $B$ -complex plane, given by  $\Re(B) < -a_{\max} - 2$ , where  $a_{\max}$  denotes the maximal power of  $r$  in the behaviour of the source-term  $\bar{\tau}_n^{\mu\nu}$  when  $r \rightarrow +\infty$ , i.e. it corresponds to the maximal growth of the source-term at infinity (notice that  $a_{\max}$  gets larger and larger when we increase the post-Newtonian order  $n$ ). Next, we can prove that the latter function of  $B$  generates a (unique) analytic continuation down to a neighbourhood of the origin  $B = 0$ , except at  $B = 0$  itself, around which value it admits a Laurent expansion with multiple poles up to some finite order. Then, we consider the Laurent expansion of that function when  $B \rightarrow 0$  and compute the finite part ( $\mathcal{FP}$ ), or coefficient of the zero-th power of  $B$ , of that expansion. This defines our generalized Poisson integral:

$$\widetilde{\Delta^{-1}} [\bar{\tau}_n^{\mu\nu}] = \mathcal{FP} \Delta^{-1} [r^B \bar{\tau}_n^{\mu\nu}]. \quad (3)$$

The resulting “generalized” Poisson integral constitutes an appropriate solution of the post-Newtonian equation. However this is only a *particular* solution of the Poisson equation, and the most general solution will be the sum of that particular solution and the most general solution of the corresponding homogeneous equation. At this stage, considering the post-Newtonian iteration scheme alone, we cannot do more and therefore we leave the homogeneous solution unspecified (we can see *a posteriori* that it is associated with radiation-reaction effects).

The second problem has to do with the *near-zone* limitation of the post-Newtonian approximation. Indeed the post-Newtonian expansion assumes that all retardations  $r/c$  are small, so it can be viewed as a formal *near-zone* expansion when  $r \rightarrow 0$ , which is valid only in the region surrounding the source that is of small extent with respect to the typical wavelength of the emitted radiation:  $r \ll \lambda$  (if we locate the origin of the coordinates  $r = 0$  inside the source). Therefore, the fact that the coefficients of the post-Newtonian expansion blow up at spatial infinity, when  $r \rightarrow +\infty$ , has nothing to do with the actual behaviour of the field at infinity. The serious consequence is that it is not possible, *a priori*, to implement within the post-Newtonian iteration the physical information that the matter system is isolated from the rest of the universe. Most importantly, the no-incoming radiation condition, imposed at past null infinity, cannot be taken into account, *a priori*, into the scheme. In a sense the post-Newtonian approximation

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<sup>3</sup>The problem is somewhat similar to what happens in Newtonian cosmology. Here we have to solve the Poisson equation  $\Delta U = -4\pi G\rho$ , where the density  $\rho$  of the cosmological fluid is constant all over space:  $\rho = \rho(t)$ . Clearly the Poisson integral of a constant density does not make sense, as it diverges at the bound at infinity like the integral  $\int r dr$ . This nonsensical result has occasionally been referred to as the “paradox of Seeliger”. However the problem is solved once we realize that the Poisson integral does not constitute the appropriate solution of the Poisson equation in the context of Newtonian cosmology. A well-defined solution is simply given by  $U = -\frac{2}{3}\pi G\rho r^2$ .

is not “self-supporting”, because it necessitates some information taken from outside its own domain of validity.

The solution of the problem of the near-zone limitation of the post-Newtonian expansion resides in the *matching* of the near-zone field to the exterior field, a solution of the vacuum equations outside the source which has been developed in previous work [8] using some post-*Minkowskian* and multipolar expansions. In the case of post-Newtonian sources, the near zone, i.e.  $r \ll \lambda$ , covers entirely the source, because the source’s radius itself is such that  $a \ll \lambda$ . Thus the near zone overlaps with the exterior zone where the multipole expansion is valid. What we do is to impose a matching condition resulting from the numerical equality between the multipolar and post-Newtonian fields, verified in the external part of the near-zone (say  $a < r \ll \lambda$ , where  $a$  is the size of the source). The matching equation reads

$$\overline{\mathcal{M}(h^{\mu\nu})} \equiv \mathcal{M}(\overline{h}^{\mu\nu}) , \quad (4)$$

where  $\mathcal{M}$  symbolizes the multipolar (actually multipolar-post-Minkowskian) series, and as before the overbar refers to the post-Newtonian or near-zone expansion. We emphasize that the matching equation is physically justified only for post-Newtonian sources, for which the exterior near-zone exists.

The requirement of matching to the post-Newtonian solution has been shown (in a previous work [10]) to entirely determine, up to any post-Newtonian order, the multipole moments parametrizing the exterior field. In our more recent paper [2] we have proved that the still undetermined homogeneous solutions in the post-Newtonian iteration alluded to before are also fully determined by the matching equation. These homogeneous solutions are associated with radiation-reaction effects — for instance they incorporate the dominant radiation-reaction force at the 2.5PN order, as well as the tail contribution in the radiation-reaction force which is known to arise at the 4PN order [11, 12, 13]). In conclusion, the post-Newtonian expansion of the field inside an isolated system is fully determined (it can be *indefinitely* reiterated, up to any post-Newtonian order) by the matching to the exterior solution satisfying the correct boundary condition at infinity — notably the absence of incoming radiation from past null infinity.

### 3 Dynamics and radiation of compact binaries

By equations of motion we mean the explicit expression of the accelerations of the compact bodies in terms of the positions and velocities. In Newtonian gravity, writing the equations of motion for a system of  $N$  particles is trivial; in general relativity, even writing the equations in the case  $N = 2$  is difficult. The first relativistic term, at the 1PN order, was derived by Einstein, Infeld and Hoffmann [14] by means of their famous “surface-integral” method, by which the equations of motion are deduced from the *vacuum* field equations, and so which is applicable to any compact objects (they be neutron stars, black holes, or, maybe, naked singularities).

Concerning the 2PN and 2.5PN approximations the result for the equations of binary motion in harmonic coordinates was obtained by Damour and Deruelle and collaborators [18, 15, 16, 17]. The corresponding result for the ADM-Hamiltonian of two particles at the 2PN order was given in Ref. [19]. By ADM-Hamiltonian we refer to the Fokker-type Hamiltonian, which is obtained from the matter-plus-field Arnowitt-Deser-Misner Hamiltonian by eliminating the field degrees of freedom. Kopeikin [20] derived the 2.5PN equations of motion for two extended compact objects. The 2.5PN-accurate harmonic-coordinate equations as well as the complete gravitational field (namely the metric  $g_{\mu\nu}$ ) were computed by Blanchet, Faye and Ponsot [21], and by Pati and Will [22].

It is important to realize that the 2.5PN equations of motion are known to hold in the case of binary systems of strongly self-gravitating bodies. This is *via* an “effacing” principle for the internal structure of the bodies. As a result, the equations depend only on the “Schwarzschild” masses,  $m_1$  and  $m_2$ , of the compact objects, as has been explicitly verified up to the 2.5PN order [20, 23]. The 2.5PN equations of motion have also been established by Itoh, Futamase and Asada [24, 25], who use a variant of the surface-integral approach [14], valid for compact bodies.

The present state of the art is the 3PN order. The equations of motion at this order have been worked out independently by two groups, by means of different methods, and with equivalent results. On one hand, Jaranowski and Schäfer [26, 27], and Damour, Jaranowski and Schäfer [28, 29, 30], employ the ADM-Hamiltonian formalism of general relativity; on the other hand, Blanchet and Faye [31, 34, 32, 33],

de Andrade, Blanchet and Faye [35], and Blanchet and Iyer [36], founding their approach on the post-Newtonian iteration initiated in Ref. [21], compute directly the equations of motion from which they infer the Lagrangian (instead of the Hamiltonian) in harmonic coordinates. The end results have been shown [29, 35] to be physically equivalent in the sense that there exists a unique “contact” transformation of the dynamical variables, that changes the harmonic-coordinates Lagrangian obtained in Ref. [35] into a new Lagrangian, whose Legendre transform coincides exactly with the Hamiltonian given in Ref. [28]. The 3PN equations of motion, however, depend on one unspecified numerical coefficient,  $\omega_{\text{static}}$  in the ADM-Hamiltonian formalism and  $\lambda$  in the harmonic-coordinates approach, which is due to some incompleteness of the Hadamard self-field regularization method. This coefficient has been fixed by means of a dimensional regularization in Ref. [30] (see Section 5).

So far the status of the post-Newtonian equations of motion is quite satisfying. There is mutual agreement between all the results obtained by means of different approaches and techniques, whenever it is possible to compare them: point-particles described by Dirac delta-functions, extended post-Newtonian fluids, surface-integrals methods, mixed post-Minkowskian and post-Newtonian expansions, direct post-Newtonian iteration and matching, harmonic coordinates versus ADM-type coordinates, different processes or variants of the regularization of the self field of point-particles.

Let us remark that the 3PN equations of motion are merely “Newtonian” as regards the radiative aspects of the problem, because with that precision they contain the radiation reaction force at only the lowest 2.5PN order. From the conservative part of the 3PN dynamics (neglecting the 2.5PN radiation reaction) we shall obtain the binary’s center-of-mass energy  $E$  at the 3PN order. We now want to compute the variation of  $E$  because of the emission of gravitational radiation. For this purpose we replace the knowledge of the radiation reaction force in the local equations of motion by the computation of the total energy flux in gravitational waves, say  $\mathcal{L}$ , and we apply the energy balance equation

$$\frac{dE}{dt} = -\mathcal{L} . \quad (5)$$

Therefore in our approach the computation of the center-of-mass energy  $E$  constitutes only one “half” of the solution of the problem. The second “half”, that of the computation of the energy flux  $\mathcal{L}$ , is to be carried out by application of a wave-generation formalism.

Following earliest computations at the 1PN level [37, 38] (at a time when the post-Newtonian corrections in  $\mathcal{L}$  had a purely academic interest), the energy flux of inspiralling compact binaries was completed to the 2PN order by Blanchet, Damour and Iyer [39], using the 2PN wave-generation formalism of Ref. [9], and, independently, by Will and Wiseman [40], using their own formalism (see Refs. [41, 42] for joint reports of these calculations). The preceding approximation, 1.5PN, which represents in fact the dominant contribution of tails in the wave zone, had been obtained in Refs. [43, 44] by application of a formula for tail integrals given in Ref. [45]. Higher-order tail effects at the 2.5PN and 3.5PN orders, as well as a crucial contribution of tails generated by the tails themselves (the so-called “tails of tails”) at the 3PN order, were obtained by Blanchet [46, 47]. However, unlike the 1.5PN, 2.5PN and 3.5PN orders that are entirely composed of tail terms, the 3PN approximation involves also, besides the tails of tails, many non-tail contributions coming from the relativistic corrections in the (source) multipole moments of the binary. These have been completed by Blanchet, Iyer and Joguet [48, 49], based on the expressions for the multipole moments given in Ref. [10], except for one single unknown numerical coefficient, due to the use of the Hadamard regularization, which is a combination of the parameter  $\lambda$  in the equations of motion, and a new parameter  $\theta$  coming from the computation of the 3PN quadrupole moment.

The post-Newtonian flux  $\mathcal{L}$  is in complete agreement, up to the 3.5PN order, with the result given by the very different technique of linear black-hole perturbations, valid in the “test-mass” limit where the mass of one of the bodies tends to zero (limit  $\nu \rightarrow 0$ , where  $\nu = \mu/m$ ). Linear black-hole perturbations, triggered by the geodesic motion of a small mass around the black hole, have been applied to this problem [50, 51]. This technique has culminated with the beautiful analytical methods of Sasaki, Tagoshi and Tanaka [52, 53, 54], who solved the problem up to the extremely high 5.5PN order.

## 4 Post-Newtonian templates for compact binary inspiral

The orbital phase of the binary — i.e. the integral of the frequency  $\omega(t)$ :

$$\Phi(t) = \int \omega(t) dt, \quad (6)$$

constitutes the crucial quantity to be monitored (and therefore to be predicted) in the detectors. The templates built from the theoretical prediction for the orbital phase should be accurate enough over most of the inspiral phase, within the frequency bandwidth of the detectors, with reduced cumulative phase lags, so that the phasing errors are not significant when one attempts to extract the values of the binary's parameters (essentially the masses and spins) from the data.

The relevant model for describing the inspiral phase consists of two point-masses moving under their mutual gravitational attraction. As a simplification for the theoretical analysis, the orbit of inspiralling binaries can be considered to be circular, apart from the gradual inspiral, with a good approximation. The templates are based on the energy-balance equation (5), from which one deduces the orbital phase as

$$\Phi_c - \Phi = \int_{\omega_c}^{\omega} \frac{\omega dE}{\mathcal{L}}, \quad (7)$$

where  $\Phi_c$  and  $\omega_c$  denote the values at the instant of coalescence. The number of gravitational-wave cycles left from the current time till the coalescence instant (we consider only the dominant harmonics at twice the orbital frequency) is

$$\mathcal{N} = \frac{\Phi_c - \Phi}{\pi}. \quad (8)$$

It is clear — because of Eq. (1) — that  $\mathcal{N}$  is of the order of the inverse of radiation-reaction effects, hence the formal “post-Newtonian” order is  $\mathcal{N} = \mathcal{O}(c^{+5})$ .

As a matter of fact,  $\mathcal{N}$  will be a large number, approximately equal to  $1.6 \times 10^4$  in the case of two neutron stars between 10 and 1000 Hz (roughly the frequency bandwidth of the detector VIRGO). Data analysts have estimated that, in order not to suffer a too severe reduction of signal-to-noise ratio, one should monitor the phase evolution with an accuracy comparable to one gravitational-wave cycle (i.e.  $\delta\mathcal{N} \sim 1$ ), over the whole detector's bandwidth. From a strict post-Newtonian point of view, we see that the consequence, since the “Newtonian” number of cycles is formally  $\mathcal{O}(c^{+5})$ , is that any post-Newtonian correction therein that is larger than the order  $c^{-5}$  is expected to contribute to the phase evolution more than what is allowed by the previous estimate. Therefore, one expects that in order to construct accurate enough templates it will be necessary to compute the orbital phase up to at least the 2.5PN order at a minimum. This back-on-the-envelope estimate has been confirmed by measurement-accuracy analyses [56, 51, 57, 58] which showed that in advanced generations of detectors the 3PN approximation (or, even better, the 3.5PN one) is required in the case of inspiralling neutron star binaries.

The first ingredient in the theoretical calculation is the binding energy  $E$  (in the center-of-mass frame), defined as being the invariant energy associated with the *conservative* part of the binary's 3PN dynamics (we ignore the radiation reaction effect at the 2.5PN order). The center-of-mass energy  $E$  is deduced from the Hamiltonian in ADM-type coordinates [26, 27, 28, 29, 30], or equivalently from the Lagrangian in harmonic-coordinates [31, 32, 33, 34, 35, 36]. Restricting our consideration to circular orbits, the energy is a function of a single variable, the radial distance  $r$  between the two particles in a given coordinate system. In fact it is better to express the energy in terms of the frequency  $\omega = \frac{2\pi}{P}$  of the orbital motion, or, rather, in terms of the particular frequency-related parameter

$$x \equiv \left( \frac{GM\omega}{c^3} \right)^{2/3}. \quad (9)$$

The individual masses of the black holes are denoted by  $m_1$  and  $m_2$ , and the total mass is  $M = m_1 + m_2$ . The interest of using the parameter  $x$ , instead of some coordinate distance  $r$ , is that the energy function  $E(x)$  then takes an invariant invariant (the same in different coordinate systems). The result we get consists of the Newtonian contribution, proportional to  $x$ , followed by post-Newtonian corrections up to the 3PN order:

$$\begin{aligned}
E = & -\frac{\mu c^2 x}{2} \left\{ 1 + \left( -\frac{3}{4} - \frac{1}{12} \nu \right) x + \left( -\frac{27}{8} + \frac{19}{8} \nu - \frac{1}{24} \nu^2 \right) x^2 \right. \\
& + \left. \left( -\frac{675}{64} + \left[ \frac{209323}{4032} - \frac{205}{96} \pi^2 - \frac{110}{9} \lambda \right] \nu - \frac{155}{96} \nu^2 - \frac{35}{5184} \nu^3 \right) x^3 \right\}. \quad (10)
\end{aligned}$$

This expression involves the useful ratio between reduced and total masses:

$$\nu = \frac{\mu}{M} \quad \text{where} \quad \mu = \frac{m_1 m_2}{M}. \quad (11)$$

This ratio is interesting because of its range of variation:  $0 < \nu \leq \frac{1}{4}$ , where  $\nu = \frac{1}{4}$  in the equal-mass case and  $\nu \rightarrow 0$  in the test-mass limit for one of the bodies. The parameter  $\lambda$  denotes a point-mass regularization-constant and will be discussed in Section 5.

The second ingredient in this analysis concerns the gravitational-wave luminosity  $\mathcal{L}$ , calculated, in the post-Newtonian approximation, from a wave-generation formalism valid for extended “fluid” systems [9, 47, 10], and then specialized to binary systems of point-particles [39, 46, 47, 48, 49]. The calculation takes properly into account the relativistic corrections linked with the description of the source (multipole moments), as well as the non-linear effects in the propagation of the waves from the source to the far zone. The crucial input of any post-Newtonian computation of the flux is the mass quadrupole moment (because the post-Newtonian precision required for the higher multipole moments is smaller). The 3PN quadrupole moment for circular binary orbits in a harmonic coordinate system is of the form<sup>4</sup>

$$I_{ij} = \mu \left( A \hat{x}_{ij} + B \frac{r^2}{c^2} \hat{v}_{ij} \right), \quad (12)$$

and has been obtained recently by Blanchet, Iyer and Joguet [48], with result

$$\begin{aligned}
A = & 1 + \left( -\frac{1}{42} - \frac{13}{14} \nu \right) \gamma + \left( -\frac{461}{1512} - \frac{18395}{1512} \nu - \frac{241}{1512} \nu^2 \right) \gamma^2 \\
& + \left( \frac{395899}{13200} - \frac{428}{105} \ln \left( \frac{r}{r_0} \right) + \left[ \frac{139675}{33264} - \frac{44}{3} \ln \left( \frac{r}{r'_0} \right) - \frac{44}{3} \xi - \frac{88}{3} \kappa \right] \nu + \frac{162539}{16632} \nu^2 + \frac{2351}{33264} \nu^3 \right) \gamma^3, \quad (13)
\end{aligned}$$

$$\begin{aligned}
B = & \frac{11}{21} - \frac{11}{7} \nu + \left( \frac{1607}{378} - \frac{1681}{378} \nu + \frac{229}{378} \nu^2 \right) \gamma \\
& + \left( -\frac{357761}{19800} + \frac{428}{105} \ln \left( \frac{r}{r_0} \right) + \left[ -\frac{75091}{5544} + \frac{44}{3} \zeta \right] \nu + \frac{35759}{924} \nu^2 + \frac{457}{5544} \nu^3 \right) \gamma^2, \quad (14)
\end{aligned}$$

where the post-Newtonian ordering parameter (defined in harmonic coordinates) reads

$$\gamma = \frac{GM}{rc^2}. \quad (15)$$

Notice the logarithms entering these formulas at the 3PN order. One type involves a constant length scale  $r_0$  coming from the general wave-generation formalism of Refs. [9, 47, 10], and which corresponds to some “infrared” cut-off in the bound at infinity of the integrals defining the multipole moments. The constant  $r_0$  should and will cancel out when considering the complete multipole expansion of the field exterior to the source. The other type of logarithm contains a different length scale  $r'_0$ , which should rather be viewed as an “ultra-violet” cut-off associated with the singular behaviour of the metric at the location of the point-particles;  $r'_0$  is related to the two constants  $r'_1$  and  $r'_2$  — one for each particles — appearing in the 3PN equations of motion of Refs. [31, 34, 32, 33, 35, 36] by

$$\ln r'_0 = \frac{m_1}{M} \ln r'_1 + \frac{m_2}{M} \ln r'_2. \quad (16)$$

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<sup>4</sup>We neglect a 2.5PN term and denote e.g.  $\hat{x}_{ij} = x_i x_j - \frac{1}{3} \delta_{ij} r^2$ , where  $x_i$  is the harmonic-coordinate separation between the two bodies and  $r = |\mathbf{x}|$ .

As we know that  $r'_1$  and  $r'_2$  are gauge constants, i.e. they can be eliminated by a gauge transformation at the 3PN order,  $r'_0$  will necessarily disappear from our physical, gauge-invariant, result at the end. Besides the harmless constants  $r_0$  and  $r'_0$ , there are three point-mass regularization-constants in Eqs. (13)-(14):  $\xi$ ,  $\kappa$  and  $\zeta$  (see Section 5). However, we shall see that the energy flux for circular orbits depends on one combination only of these constants:  $\theta = \xi + 2\kappa + \zeta$ .

Through 3.5PN order, the result is decomposed into an “instantaneous” part, i.e. generated solely by the multipole moments of the source, and a tail part. What we call here the tail part is in fact a complicated sum of “tails”, “tail squares”, and “tails of tails”, as computed in Ref. [47]. We find, respectively,

$$\begin{aligned}\mathcal{L}_{\text{inst}} &= \frac{32c^5}{5G}\nu^2\gamma^5\left\{1 + \left(-\frac{2927}{336} - \frac{5}{4}\nu\right)\gamma + \left(\frac{293383}{9072} + \frac{380}{9}\nu\right)\gamma^2\right. \\ &+ \left(\frac{53712289}{1108800} - \frac{1712}{105}\ln\left(\frac{r}{r_0}\right)\right. \\ &\left. + \left[-\frac{332051}{720} + \frac{123}{64}\pi^2 + \frac{110}{3}\ln\left(\frac{r}{r'_0}\right) + 44\lambda - \frac{88}{3}\theta\right]\nu - \frac{383}{9}\nu^2\right)\gamma^3\Big\},\end{aligned}\quad (17)$$

$$\begin{aligned}\mathcal{L}_{\text{tail}} &= \frac{32c^5}{5G}\nu^2\gamma^5\left\{4\pi\gamma^{3/2} + \left(-\frac{25663}{672} - \frac{109}{8}\nu\right)\pi\gamma^{5/2}\right. \\ &+ \left(-\frac{116761}{3675} + \frac{16}{3}\pi^2 - \frac{1712}{105}C - \frac{856}{105}\ln(16\gamma) + \frac{1712}{105}\ln\left(\frac{r}{r_0}\right)\right)\gamma^3 \\ &\left. + \left(\frac{90205}{576} + \frac{3772673}{12096}\nu + \frac{32147}{3024}\nu^2\right)\pi\gamma^{7/2}\right\}.\end{aligned}\quad (18)$$

The Newtonian result has been factorized out in front. Here  $C = 0.577\dots$  denotes the Euler constant. As we can see, the constant  $r_0$  drops out from the sum of Eqs. (17) and (18). However, the gauge constant  $r'_0$  does not seem to disappear at this stage, but that is simply due to our use of the post-Newtonian parameter  $\gamma$  defined by (15), and which depends *via* the equation of motion on the choice of harmonic coordinates. After substituting the frequency-related parameter  $x$  in place of  $\gamma$  (with the help of the 3PN equations of motion), we find that  $r'_0$  does cancel as well — this nicely illustrates the consistency between our two computations, in harmonic-coordinates, of the equation of motion on one hand and the multipole moments on the other hand. Finally we obtain

$$\begin{aligned}\mathcal{L} &= \frac{32c^5}{5G}\nu^2x^5\left\{1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2}\right. \\ &+ \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{535}{24}\nu\right)\pi x^{5/2} \\ &+ \left(\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}C - \frac{856}{105}\ln(16x)\right. \\ &+ \left[-\frac{11497453}{272160} + \frac{41}{48}\pi^2 + \frac{176}{9}\lambda - \frac{88}{3}\theta\right]\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\Big)x^3 \\ &\left. + \left(-\frac{16285}{504} + \frac{176419}{1512}\nu + \frac{19897}{378}\nu^2\right)\pi x^{7/2}\right\}.\end{aligned}\quad (19)$$

The last test (but not the least) is that Eq. (19) is in perfect agreement, in the test-mass limit  $\nu \rightarrow 0$ , with the result of linear black-hole perturbations obtained by Tagoshi and Sasaki [53].

## 5 Problem of regularization ambiguities

A model of structureless point masses is expected to be sufficient to describe the inspiral phase of compact binaries. Thus we want to compute the metric (and its gradient needed in the equations of motion) at



the 3PN order for a system of two point-like particles. Applying general expressions for the metric valid in the case of continuous (smooth) matter sources, we find that most of the integrals become divergent at the location of the particles. Consequently we must supplement the calculation by a prescription for how to remove (i.e. to regularize) the infinite part of these integrals.

The “standard” Hadamard regularization yields some ambiguous results for the computation of certain integrals at the 3PN order, as Jaranowski and Schäfer [26, 27] noticed in their computation of the equations of motion within the ADM-Hamiltonian formulation of general relativity. They showed that there are *two* and *only two* types of ambiguous terms in the 3PN Hamiltonian, which were then parametrized by two unknown numerical coefficients  $\omega_{\text{static}}$  and  $\omega_{\text{kinetic}}$ .

Blanchet and Faye [32, 33], motivated by the previous result, introduced their “improved” Hadamard regularization, based on a theory of pseudo-functions and generalized distributional derivatives. This new regularization is mathematically well-defined and free of ambiguities; in particular it yields unique results for the computation of any of the integrals occurring in the 3PN equations of motion. Unfortunately, this regularization turned out to be in a sense incomplete, because it was found [31, 34] that the 3PN equations of motion involve *one* and *only one* unknown numerical constant, called  $\lambda$ , which cannot be determined within the method. The comparison of this result with the work of Jaranowski and Schäfer [26, 27], on the basis of the computation of the invariant energy of binaries moving on circular orbits, showed [31] that

$$\omega_{\text{kinetic}} = \frac{41}{24}, \quad (20)$$

$$\omega_{\text{static}} = -\frac{11}{3}\lambda - \frac{1987}{840}. \quad (21)$$

Therefore, the ambiguity  $\omega_{\text{kinetic}}$  is fixed, while  $\lambda$  is equivalent to the other ambiguity  $\omega_{\text{static}}$ . The value of  $\omega_{\text{kinetic}}$  given by Eq. (20) was recovered by Damour, Jaranowski and Schäfer [28], who proved that this value is the unique one for which the global Poincaré invariance of their formalism is verified. By contrast, the harmonic-coordinate conditions preserve the Poincaré invariance, and therefore the associated equations of motion should be Lorentz-invariant, as was indeed found to be the case by Blanchet and Faye [31, 34], thanks in particular to their use of a Lorentz-invariant regularization [33] (hence their determination of  $\omega_{\text{kinetic}}$ ).

The appearance of one and only one physical indeterminacy  $\lambda \Leftrightarrow \omega_{\text{static}}$  in the equations of motion constitutes a quite striking fact, specifically related to the use of an Hadamard-type regularization. Mathematically speaking, the presence of  $\lambda$  is (probably) related to the fact that it is impossible to construct a distributional derivative operator satisfying the Leibniz rule for the derivation of the product. The Einstein field equations can be written into many different forms, by operating some terms by parts with the help of the Leibniz rule. All these forms are equivalent in the case of regular sources, but they become inequivalent for point particles if the derivative operator violates the Leibniz rule.

In Ref. [60] it has been argued that the numerical value of the parameter  $\omega_{\text{static}}$  could be  $\simeq -9$ , because for such a value some different resummation techniques, *viz* Padé approximants [58] and effective-one-body (EOB) method [59], when they are implemented at the 3PN order, give approximately the same numerical result for the location of the last stable circular orbit. Even more, it was suggested [60] that  $\omega_{\text{static}}$  might be precisely equal to  $\omega_{\text{static}}^*$ , with

$$\omega_{\text{static}}^* = -\frac{47}{3} + \frac{41}{64}\pi^2. \quad (22)$$

(We have  $\omega_{\text{static}}^* = -9.34\dots$ .) However, the value of  $\omega_{\text{static}}$  in general relativity has been computed by Damour, Jaranowski and Schäfer [30] by means of a dimensional regularization, instead of an Hadamard-type one, within the ADM-Hamiltonian formalism. Their result is

$$\omega_{\text{static}} = 0 \iff \lambda = -\frac{1987}{3080}. \quad (23)$$

As Damour *et al* [30] argue, clearing up the ambiguity is made possible by the fact that the dimensional regularization, contrary to the Hadamard regularization, respects all the basic properties of the algebraic

and differential calculus of ordinary functions<sup>5</sup>.

Let us comment that the use of a self-field regularization in this problem, it be dimensional or based on the Hadamard *partie finie*, signals a somewhat unsatisfactory situation on the physical point of view, because we would like to perform, ideally, a complete calculation valid for extended bodies, taking into account the details of the internal structure of the bodies (energy density, pressure, internal velocities, etc.). By considering the limit where the radii of the objects tend to zero, one should recover the same result as obtained by means of the point-mass regularization, and determine the value of the regularization parameter  $\lambda$ . Because of considerable difficulties arising at the 3PN order this program has not yet been achieved.

Concerning the 3PN radiation field of two point masses — the second half of the problem, besides the 3PN equations of motion —, Blanchet, Iyer and Joguet [48] used the (standard) Hadamard regularization and found necessary to introduce three additional regularization constants  $\xi$ ,  $\kappa$  and  $\zeta$ . However the total gravitational-wave flux, in the case of circular orbits, depends only on the linear combination

$$\theta = \xi + 2\kappa + \zeta. \quad (24)$$

Furthermore, this  $\theta$  comes in at the same level as  $\lambda$ , so there is in fact only one unknown constant in the flux [see Eqs. (17)-(19)]. Notice that the improved version of the Hadamard regularization proposed in Refs. [32, 33] should be able, in principle, to fix the value of the constant  $\zeta$ .

## 6 Accuracy of the post-Newtonian approximation

In this section we discuss (some aspects of) the accuracy of the post-Newtonian approximation as regards the determination of the binary’s innermost circular orbit (ICO). The ICO will be defined by the minimum, when it exists, of the energy function  $E(x)$  given by Eq. (10). Notice that we do not define the ICO as an ISCO, i.e. as a point of dynamical general-relativistic instability. See Ref. [36] for a discussion of the dynamical stability of circular binary orbits at the 3PN order.

Let us first confront the prediction of the standard (Taylor-based) post-Newtonian approximation at the 3PN order for the ICO — as given by the minimum of Eq. (10) — with a recent numerical calculation by Gourgoulhon, Grandclément and Bonazzola [61, 62]. These authors obtained numerically the energy  $E(\omega)$  of binary black holes along evolutionary sequences of equilibrium configurations under the assumptions of conformal flatness for the spatial metric and of exactly circular orbits. The latter restriction is implemented by requiring the existence of an “helical” Killing vector, time-like inside the light cylinder associated with the circular motion and space-like outside. The numerical calculation [61, 62] has been performed in the case of *corotating* black holes, which are spinning with the orbital angular velocity  $\omega$ . For the comparison we must therefore include within the post-Newtonian formalism yielding Eq. (10) the effects of spins appropriate to two Kerr black holes rotating at the orbital rate  $\omega$ .

The Figure 1 (issued from Ref. [63]) presents the post-Newtonian results for  $E_{\text{ICO}}$  in the case of irrotational and corotational binaries. The points indicated by 1PN, 2PN and 3PN are defined from the obvious truncation of Eq. (10). The points  $1\text{PN}^{\text{corot}}$ ,  $2\text{PN}^{\text{corot}}$  and  $3\text{PN}^{\text{corot}}$  take into account the spin effects of corotational binaries and are computed in Ref. [63]. Notice that the irrotational and corotational configurations differ only from the 2PN order. As we can see the 3PN points, and even the 2PN ones, are rather close to the numerical value (marked by an asterisk in Figure 1). As expected, the best agreement is for the 3PN approximation and in the case of corotation: i.e. the point  $3\text{PN}^{\text{corot}}$ . So our first conclusion is that the location of the ICO as computed by numerical relativity, under the helical-symmetry approximation [61, 62], is in good agreement with post-Newtonian predictions. This constitutes an appreciable improvement of the previous situation, because we recall that the earlier estimates of the ICO in post-Newtonian theory [64] and numerical relativity [65, 66] strongly disagree with each other, and do not match with the present 3PN results (see Ref. [62] for further discussion).

Our second conclusion comes from the fact that the 2PN and 3PN values in Figure 1 — either for irrotational or corotational binaries — are so close to each other. Indeed it seems that the 3PN points

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<sup>5</sup>Note that the result (23) is very different from  $\omega_{\text{static}}^*$  given by Eq. (22): this suggests that the resummation techniques (Padé approximants and EOB method), though they are designed to “accelerate” the convergence of the post-Newtonian series, do *not* in fact converge toward the same exact solution (or, at least, not as fast as expected). See Section 7 for discussion on this point.

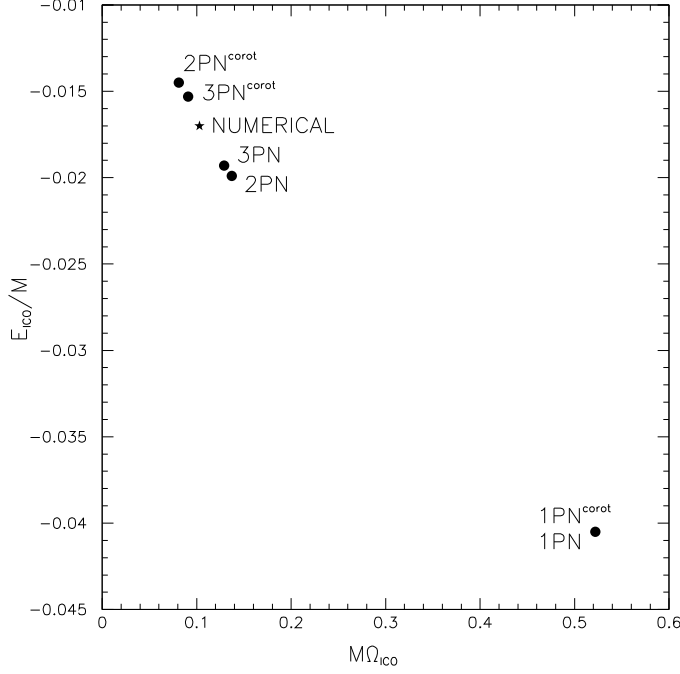


Figure 1: The center-of-mass energy  $E_{\text{ICO}}$  versus  $\omega_{\text{ICO}}$  in the equal-mass case ( $\nu = \frac{1}{4}$ ). The asterisk marks the result calculated by numerical relativity [61, 62]. The points indicated by 1PN, 2PN and 3PN correspond to irrotational binaries, while the points denoted by 1PN<sup>corot</sup>, 2PN<sup>corot</sup> and 3PN<sup>corot</sup> describe corotational binaries. Both 3PN are 3PN<sup>corot</sup> are shown for  $\omega_{\text{static}} = 0$ .

are useful only to *confirm* the result given by the 2PN ones. This is a quite satisfying state of affairs because it indicates that the post-Newtonian approximation converges very well (in the sense of Cauchy’s criterion). Figure 1 shows that the post-Newtonian approximation is well qualified to accurately locate the ICO (of course for this purpose one must go to the 3PN order — the 1PN approximation is clearly not accurate enough).

Let us elaborate more on this point. First we make a few order-of-magnitude estimates. At the location of the ICO we find (see Figure 1) that the frequency-related parameter  $x$  defined by Eq. (9) is approximately of the order of 20%. Therefore, we might *a priori* expect that the contribution of the 1PN approximation to the energy at the point of the ICO should be of the order of 20%. For the present discussion we take the pessimistic view that the order of magnitude of an approximation represents also the order of magnitude of the higher-order terms that are neglected. We see that the 1PN approximation should yield a rather poor estimate of the “exact” result, but this is quite normal at this very relativistic point where the orbital velocity is  $\frac{v}{c} \sim \sqrt{x} \sim 50\%$ . By the same argument we infer that the 2PN approximation should do much better, with fractional errors of the order of  $x^2 \sim 5\%$ , while the 3PN approximation will be even better, with the precision  $x^3 \sim 1\%$ .

The simple order-of-magnitude estimate suggests therefore that the 3PN order should be close to the “exact” solution for the ICO to within 1% of fractional accuracy. We think that this is very good, and we should even remember that this estimate is pessimistic, because we could reasonably expect that the neglected higher-order approximations, 4PN and so on, are in fact much smaller numerically (e.g. of the order of  $x^4 \sim 0.2\%$ ). But let us keep for the present discussion the 1% guess for the accuracy of the 3PN approximation.

Now the previous estimates make sense only if the numerical values of the post-Newtonian coefficients in Eq. (10) are roughly of the order of one. If this is not the case, and if the coefficients increase

dangerously with the post-Newtonian order  $n$ , one sees that the post-Newtonian approximation might in fact be very bad. So let us look at the values of the coefficients of the 1PN, 2PN and 3PN approximations in Eq. (10), say

$$a_1(\nu) = -\frac{3}{4} - \frac{\nu}{12}, \quad (25)$$

$$a_2(\nu) = -\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}, \quad (26)$$

$$a_3(\nu) = -\frac{675}{64} + \left[ \frac{209323}{4032} - \frac{205}{96}\pi^2 - \frac{110}{9}\lambda \right] \nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3. \quad (27)$$

We present in Table 1 the values of these coefficients in the test-mass limit  $\nu = 0$ , and in the equal-mass case  $\nu = \frac{1}{4}$  when the ambiguity parameter takes the “uncorrect” value  $\omega_{\text{static}}^*$  [see Eq. (22)] and the correct one  $\omega_{\text{static}} = 0$  predicted by general relativity [30].

		Newtonian	$a_1(\nu)$	$a_2(\nu)$	$a_3(\nu)$
$\nu = 0$		1	-0.75	-3.37	-10.55
$\nu = \frac{1}{4}$	$\omega_{\text{static}}^* \simeq -9.34$	1	-0.77	-2.78	-8.75
$\nu = \frac{1}{4}$	$\omega_{\text{static}} = 0$	1	-0.77	-2.78	-0.97

Table 1: Sequence of coefficients of the post-Newtonian series composing the energy function (25)-(27).

Our first comment is that when  $\nu = 0$  there is an *increase* of the coefficients by roughly a factor 3 at each step. This behaviour is fairly easy to understand. It comes from the existence in the Schwarzschild metric of the famous light-ring orbit: a geodesics of photon which is a circular orbit located at  $R = 3M$  in Schwarzschild coordinates. As a result, the energy of the test particle in the Schwarzschild metric exhibits a singularity at  $x_{\text{light-ring}} = \frac{1}{3}$ ,

$$E^{\text{test}}(x) = \mu c^2 \left( \frac{1-2x}{\sqrt{1-3x}} - 1 \right). \quad (28)$$

The consequence is that the radius of convergence of the post-Newtonian series is  $\frac{1}{3}$ , so the post-Newtonian coefficients increase by a factor  $\sim 3$ . So the post-Newtonian series is not very accurate in the case  $\nu = 0$ . This fact has motivated several statements in the literature (see e.g. [55, 56, 57, 58]), according to which the post-Newtonian approximation would be “poorly convergent”, or that there should be a “fundamental breakdown” of its validity in the regime of the ICO. This is indeed true in the *test-mass* limit ( $\nu \rightarrow 0$ ), where the post-Newtonian series converges slowly<sup>6</sup>.

On the other hand, what happens in the equal-mass case ( $\nu = \frac{1}{4}$ )? When  $\nu = \frac{1}{4}$  and we have the value  $\omega_{\text{static}}^*$  we notice that the coefficients increase approximately like in the test-mass case  $\nu = 0$ . This indicates that the gravitational interaction in the case of  $\omega_{\text{static}}^*$  looks like that in a one-body problem. We shall say that when the post-Newtonian coefficients rapidly increase with the order of approximation, the interaction is “Schwarzschild-like”.

Now when  $\nu = \frac{1}{4}$  and the ambiguity parameter takes the correct value  $\omega_{\text{static}} = 0$ , we see that the 3PN coefficient  $a_3(\frac{1}{4})$  is of the order of minus one instead of  $\sim -10$ . We think that this strongly suggests, unless 3PN happens to be quite accidental, that the post-Newtonian coefficients in general relativity do not increase very much with  $n$ , and stay rather of the order of one. This is interesting as it indicates that the actual general-relativistic two-body interaction is *not* Schwarzschild-like.

It is impossible of course to be very confident about the validity of the previous statement because we know only the coefficients up to the 3PN order. Any tentative conclusion based on the 3PN order

<sup>6</sup>Let us remark that this negative conclusion does not matter: indeed we shall never use the post-Newtonian approximation in the case  $\nu \rightarrow 0$  simply because we know the exact result which is given by Eq. (10). The exact result for the radiation field is known as well, albeit numerically only [51, 57]. Therefore we should not worry too much about the poor convergence of the post-Newtonian series in the test-mass limit. The post-Newtonian method is useless and even one might say irrelevant when considering the motion of a test particle around a Schwarzschild black hole.

can be “falsified” when we obtain the next 4PN order. Nevertheless, we feel that the mere fact that  $a_3(\frac{1}{4}) = -0.97$  in Table 1 is sufficient to motivate our (tentative) conclusion that the field of two bodies is more complicated than the Schwarzschild space-time. This conclusion is in accordance with the present author’s respectfulness of the complexity of the Einstein field equations.

The nice consequence is that because the post-Newtonian coefficients when  $\nu = \frac{1}{4}$  stay of the order of one, the *standard* post-Newtonian approach, based on the standard Taylor approximants, is probably very accurate. The post-Newtonian series seems to “converge well”, with a “convergence radius” of the order of one<sup>7</sup>. A convincing support of this view is provided by Figure 1 itself. The order-of-magnitude estimates we did at the beginning of this Section are probably correct. In particular the 3PN order should be close to the “exact” solution even in the regime of the ICO.

It is also interesting to look at the numerical values of the post-Newtonian coefficients in the total flux  $\mathcal{L}$  obtained in Eq. (19). For this case we do not have (like for the ICO) a clear point at which we can compare with a result of numerical relativity. Furthermore things concerning the convergence of the post-Newtonian series are less clear than with the energy function. A possible reason for that could be that the flux, in contrast to the energy, includes the contribution of tails at the 1.5PN, 2.5PN and 3.5PN orders, and even of tails-of-tails at the 3PN order.

	Newtonian	$b_1(\nu)$	$b_{3/2}(\nu)$	$b_2(\nu)$	$b_{5/2}(\nu)$	$b_3(\nu)$	$b_{7/2}(\nu)$
$\nu = 0$	1	-3.71	12.57	-4.93	-38.29	128.85	-101.51
$\nu = \frac{1}{4} \quad \lambda = -\frac{1987}{3080} \quad \theta = 0$	1	-4.44	12.57	-0.10	-55.80	115.26	0.47

Table 2: Post-Newtonian coefficients in the flux function (19). The coefficient  $b_3(\nu)$  contains a log-term that we have evaluated at  $x_{\text{ICO}} \sim 0.2$ . [The value  $\theta = 0$  is taken to be indicative.]

## 7 On Schwarzschild-like templates for binary inspiral

Let us finally comment about a possible implication of our conclusion as regards the validity of the so-called post-Newtonian resummation techniques, i.e. Padé approximants [58, 60], which aim at “accelerating” the convergence of the post-Newtonian series in the pre-coalescence stage, and effective-one-body (EOB) methods [59, 60], which attempt at describing the late stage of the coalescence of two black holes. These techniques are based on the idea that the gravitational two-body interaction is a “deformation” — with  $\nu \leq \frac{1}{4}$  being the deformation parameter — of the Schwarzschild space-time.

The Padé approximants are valuable tools for giving accurate representations of functions having some singularities. In the problem at hands they would be justified if the “exact” expression of the energy [whose 3PN expansion is given by Eqs. (25)-(27)] would admit some Schwarzschild-like features, and in particular a light-ring singularity. In the Schwarzschild case the Padé series converges rapidly toward the solution [58]: the Padé constructed only from the 2PN approximation of the energy — keeping only  $a_1(0)$  and  $a_2(0)$  in Eqs. (2) — already coincide with the exact result given by Eq. (8). On the other hand, the EOB method maps the post-Newtonian two-body dynamics (at the 2PN or 3PN order) on the geodesic motion on some effective metric which happens to be a  $\nu$ -deformation of the Schwarzschild space-time. In the EOB method the effective metric looks like Schwarzschild *by definition*, and we might expect the two-body interaction to own some Schwarzschild-like features.

Our comment is that the validity of these resummation techniques (and the templates for binary inspiral built on them) is questionable, because as we have seen in Section 6 the value  $\omega_{\text{static}} = 0$  suggests that most probably the two-body interaction is not Schwarzschild-like. In particular, there does not seem to exist something like a light-ring orbit which would be a deformation of the Schwarzschild one.

To test the previous idea, let us come back to the Schwarzschild limit, where we observed that the radius of convergence of the post-Newtonian series is given by the light-ring singularity at the value  $\frac{1}{3}$ . Now the radius of convergence is given by d’Alembert’s criterion as the limit when  $n \rightarrow +\infty$  of the ratio

<sup>7</sup>Actually, the post-Newtonian series could be only asymptotic (hence divergent), but nevertheless it should give good results provided that the series is truncated near some optimal order of approximation. In this discussion we assume that the 3PN order is not too far from that optimum.

of successive coefficients, i.e.  $a_{n-1}/a_n$ . We might therefore *compute* the light-ring orbit by investigating the limit

$$x_{\text{light-ring}}^{\text{test}} = \lim_{n \rightarrow +\infty} \frac{a_{n-1}^{\text{test}}}{a_n^{\text{test}}} = \frac{1}{3} . \quad (29)$$

To test for the possible existence of a light-ring singularity in the case of comparable masses, we consider the ratio between the two highest known post-Newtonian coefficients, that are  $a_2(\nu)$  and  $a_3(\nu)$ . This ratio will give an *estimate* of the “light-ring” singularity for non-infinitesimal mass ratios,

$$x_{\text{light-ring}}(\nu) \sim \frac{a_2(\nu)}{a_3(\nu)} . \quad (30)$$

Using the values given in Table 1 we obtain for equal masses in the case of the “wrong” ambiguity parameter  $\omega_{\text{static}}^* \simeq -9.34$ ,

$$x_{\text{light-ring}}(\frac{1}{4}, \omega_{\text{static}}^*) \sim 0.32 . \quad (31)$$

As we see there seems to be in this case a (pseudo-)light-ring orbit which is a small deformation of the Schwarzschild light-ring orbit given by Eq. (29). In this Schwarzschild-like situation, we expect that Padé approximants and EOB-type methods to be appropriate.

*But*, in the case of general relativity ( $\omega_{\text{static}} = 0$ ), we obtain a drastically different result:

$$x_{\text{light-ring}}(\frac{1}{4}, G.R.) \sim 2.86 . \quad (32)$$

If we believe the correctness of this estimate we must conclude that there is in fact no notion of a light-ring orbit in the real two-body case. Or, one might say (pictorially speaking) that the light-ring orbit gets hidden inside the horizon of the final black-hole formed by coalescence. If we consider the ratio between the 1PN and 2PN coefficients (instead of between 2PN and 3PN), we get the value  $\sim 0.28$  instead of Eq. (32). So at the 2PN order the field seems to admit a light ring, while at the 3PN order it does not. This reinforces our idea that it is meaningless (with our present 3PN-based knowledge, and untill fuller information is available), to assume the existence of a light-ring singularity in the equal-mass case. Our expectation, therefore, is that the conditions under which the Padé and EOB methods should be legitimate might in fact not be fulfilled.

This doubt is confirmed by the finding of Ref. [60] (already alluded to above) that in the case of the “wrong” ambiguity parameter  $\omega_{\text{static}}^* \simeq -9.34$  the Padé approximants and the EOB method at the 3PN order give the same result for the ICO. From the previous discussion we see that this agreement is to be expected because a deformed (pseudo-)light-ring singularity seems to exist with  $\omega_{\text{static}}^*$ . By contrast, in the case of general relativity, where  $\omega_{\text{static}} = 0$ , the Padé and EOB methods give quite different results (cf the figure 2 in Ref. [60]). Another confirmation comes from the light-ring singularity which is determined from the Padé approximants at the 2PN order (see Eq. (3.22) in Ref. [58]) as  $x_{\text{light-ring}}^{\text{P-2PN}}(\frac{1}{4}) \sim 0.44$ . This value is rather close to Eq. (31) but strongly disagrees with Eq. (32). Our explanation is that the Padé series might converge toward a theory having  $\omega_{\text{static}} = \omega_{\text{static}}^*$  and therefore which is different from general relativity.

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